9   Mapping the Plane with Matrices

$$\overset{M}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}} \overset{P}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \overset{P'}{\begin{bmatrix} 7 \\ -1 \end{bmatrix}}$$

$$P = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

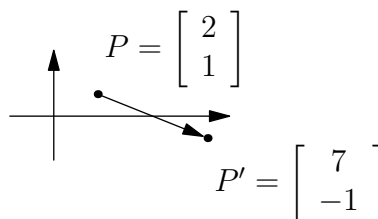$$P' = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Figure 1: Matrix multiplication is a transformation.   Figure 2: The transformation in the xy plane.

By this time, you should be comfortable with matrix multiplication. You should know what dimensional relationship needs to be true of two matrices to multiply them together.

Now we are going to consider 2×2 matrices as operators, which map points on the plane to points on the plane. That is, a 2×2 matrix maps the plane onto itself, or onto some figure (subset of the plane). This mapping is done through matrix multiplication. Because 2 coordinates determine a point in the Cartesian plane, we have the option of representing each point by a 1×2 matrix—a row vector $\begin{bmatrix} a & b \end{bmatrix}$—or a 2×1 matrix—a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$. For consistency, we will use the second format, the column vector.

Figure 1 is an example of what happens when an arbitrary matrix M operates on a point P, taking it to the point P'. The **preimage** $(2, 1)$ is mapped to the **image** $(7, -1)$ by the matrix; the geometric interpretation is shown in Figure 2. As in geometry, the preimage is the state before the transformation and the image is after.

1. (a) Use the 2×2 matrix from Figure 1 to operate on the points $(0,0)$, $(1,0)$, and $(0,1)$. What are their images? Graph them.

   (b) The preimage includes two perpendicular **unit vectors**, $(0,1)$ and $(1,0)$. What is the (i) ratio of the lengths of their images and (ii) angle between the images?

   (c) You can conclude that multiplication by matrices does not, in general, preserve which two quantities between the image and preimage?

2. (a) Now, use the 2×2 matrix from Figure 1 to operate on each of these points: $(2,1)$, $(1,0)$, $(0,-1)$ and $(-1,-2)$. Do this by consolidating all the points into one matrix, with each point as a column vector, then performing a multiplication:

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} & & & \end{bmatrix}.$$

   (b) Graph and label the preimage and the image of each point onto the same set of axes.

   (c) The points in the preimage are discontinuous, but they belong to a particular, infinite set of points. Write the equation of that set. (Hint: What is y in terms of x?)

   (d) Write an equation for the image of that set.

   (e) What other characteristic of the preimage points also applies to the image?

   (f) Name two things that seem to be conserved when mapping points with a matrix.

In Problem 2, the points of the preimage were **collinear**, as were the points of the image. You also should have noticed that the points were equally spaced in the preimage and image. Is this behavior a coincidence due to the particular matrix/set of points we picked, or is it generally true for all points and 2×2 matrices?

3. (a) Choose a different 2×2 matrix and a different set of three collinear, equally spaced unique points. Perform the appropriate matrix multiplication.

   (b) Graph and label the preimage points and the image points.

   (c) Have the collinearity and equal spacing been preserved?

   (d) Make a conjecture about when a matrix will preserve collinearity and when a matrix will preserve equal spacing.

4. Now, we will check your conjecture.

(a) Start with a general 2×2 matrix and three equally spaced points on a line, and multiply the two matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x-h & x & x+h \\ m(x-h)+k & mx+k & m(x+h)+k \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix}.$$

(b) How do you know that the second matrix indeed represents collinear and equally spaced points?

(c) Are there any sets of collinear points that aren't representable by the 2×3 matrix?

(d) Are the points in the image collinear? Show why or why not.

(e) Can you find values for a, b, c, and d so that the image does not lie on a unique line? (Hint: all of the points in the image must lie on no line, or on multiple lines.)

(f) Use the distance formula—or some other justification—to answer whether the points in the image are equally spaced.

5. There is a point which remains **fixed**—its image is the same as its preimage—when multiplied by the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. That is, $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Solve the above matrix equation for x and y to find the point.

(b) There is a point $Q = \begin{bmatrix} e \\ f \end{bmatrix}$ that remains fixed no matter what matrix you multiply it by. Can you guess what point that is?

(c) Prove your conjecture by plugging your point Q into $\begin{bmatrix} a & b \\ c & d \end{bmatrix} Q = Q$.

Matrix multiplication is a **linear transformation**. One way to think of a linear transformation is a transformation which takes lines to other lines, and keeps equally spaced points equally spaced. With this in mind, let's investigate the different kinds of matrix mappings. Recall the transformations from geometry: the identity, reflection across a line, rotation about a point, translation, and glide reflection preserve length, while others such as stretches and dilations change size. We will look for matrix representations of these, and if there are any matrix transformations new to us. We will also be investigating the case where multiple points in a preimage are mapped to the same point in the image.

6. Begin with a triangle with vertices $(5, 0)$, $(10, 0)$, and $(5, 10)$, as shown in Figure 3.

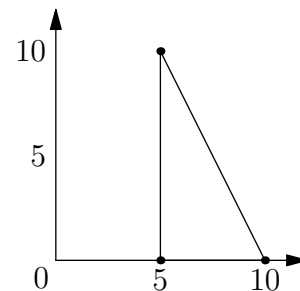(a) Map the vertices with the following matrices:

i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ iii. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

ii. $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

Figure 3: Problem 6's preimage.

(b) Why will the new triangle defined by these vertices be the image of the starting triangle?

(c) Accurately graph the preimage, then the image for each matrix on three separate sets of axes.

(d) For each, describe the transformation as fully as you can. Try to classify them on the transformations we mentioned earlier, and quantify them if necessary (e.g. to describe the line of reflection or angle of rotation).

7. Soon, we will map the unit square, which is shown in Figure 4: it has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. We could actually get the entire image from the image of the unit vectors $(1, 0)$ and $(0, 1)$, which will be useful later.

(a) How can we obtain the image of $(1, 1)$ from the images of $(1, 0)$ and $(0, 1)$?
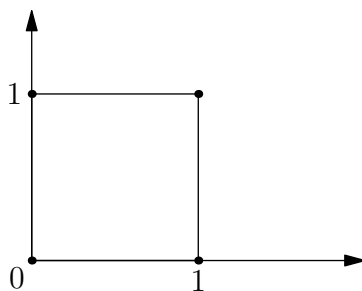
(b) Of $(0, 0)$?

Figure 4: The unit square.

8. (a) Take the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and see what it does to the unit square. Please graph this, being careful to label each point and its image. The multiplication is done for you below.

$$\begin{matrix} & & A & B & C & D & & A' & B' & C' & D' \\ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} & & & & & = & \begin{bmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

This mapping is called a **shear**[9] in the direction of the x axis, perpendicular to the y axis. Quantitatively, the preimage is sheared horizontally by a factor of 2 of its height. In this case, the square is distorted into a parallelogram by "shoving" it along the x axis without changing y. The 2 in the matrix could have been replaced by any other, nonzero[10] number and the matrix would still represent a shear in the x direction, just with a different magnitude.

(b) What happens to the area of the image versus the preimage?

(c) We have $AB = BC$, but is $A'B'$ equal to $B'C'$? Should it?

9. (a) When is the ratio of distances between points in the image the same as in the preimage?

(b) What is the image of the origin under any matrix mapping?

(c) What are the images of the points $(1,0)$ and $(0,1)$ under the mapping $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(d) Knowing the images of $(1,0)$ and $(0,1)$, how do we find the image of $(1,1)$ algebraically and geometrically?

10. How do these matrices map the plane? For each mapping, write a matrix for the images of the four corners of the unit square, then graph the preimage and image. Describe the mapping using words from geometry such as congruent, similar, rotate, reflect, shear, stretch, magnitude, and direction.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (k) $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$ (o) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (l) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (h) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (p) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(m) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (i) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (j) $\begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$ (n) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

One limitation with 2×2 matrix transformations is that they all involve a fixed point at the origin. A translation obviously takes the origin to a different point, so we can't represent it this way. One way around this problem is to do our mapping in three-dimensional space rather than the two-dimensional plane. We still keep the origin fixed, but put our preimages on the plane $z = 1$ and make sure that our images map to the same plane.

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[9]You may have heard of wind shear, which is the change of velocity of the wind with altitude. Scissors exert a shearing action on paper to cut it.

[10]If it were 0, it would become the identity transformation, which we'll talk about later.

11. Carry out the following multiplications and convince yourself they are equivalent mappings of the x and y coordinates.

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

(b) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

12. (a) Multiply these matrices: $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$.

(b) Fill in the blanks: The result of the above multiplication is that the point $(u, v, 1)$ has been translated by ___ in the x direction, ___ in the y direction, and is still anchored to the plane $z =$ ___.

13. (a) Write a matrix which translates a point $(x, y, 1)$ 4 units in the x direction and 7 units in the y direction, leaving z fixed at 1.

(b) Check your work by applying your matrix to the point $(3, 5, 1)$.

14. Do these two multiplications. What does each represent?

(a) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. What does each of these matrices represent?

(a) $\begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & -\sin\theta & \alpha \\ \sin\theta & \cos\theta & \beta \\ 0 & 0 & 1 \end{bmatrix}$

16. (a) Rewrite your translation matrix and your preimage vector from Problem 12a so that you do not restrict your translations to the plane $z = 1$, but can translate in the x, y, and z directions. (Hint: think four dimensions!)

(b) Write a matrix product that translates the point $(2, 3, -5)$ by the vector $(4, -1, 2)$.

For the curious, the translation matrix $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$ is actually two shears in 3D, as shown in Figure 5.
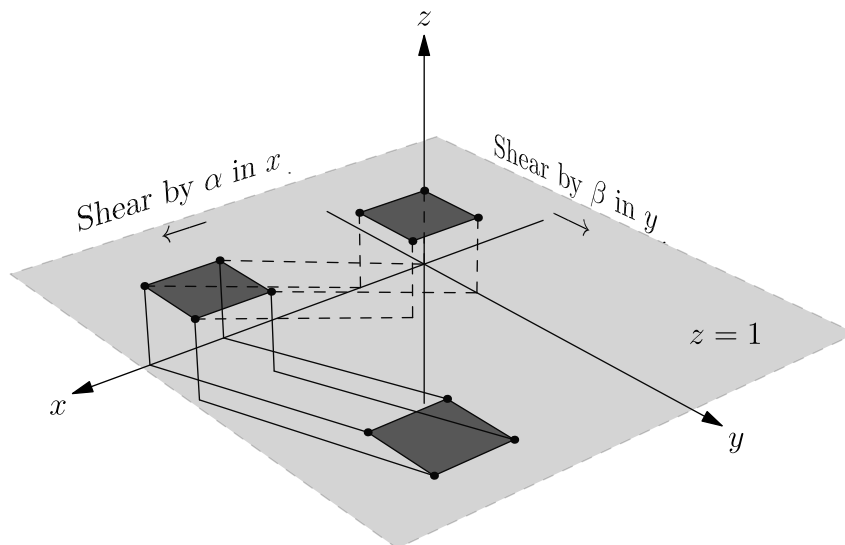

Figure 5: Translating a square by (α, β) in the plane $z = 1$. The first shear, in x, is shown in dashed lines. The second shear, in y, is shown in solid lines. The net movement is $(\alpha, \beta, 0)$.